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pale block 苍白的

(8.3)
$$(c(x_i\otimes x_j))_{i,j\in \mathbb{I}_3} = \begin{pmatrix} \varepsilon x_1\otimes x_1 & \varepsilon x_2\otimes x_1 & q_{12}\, x_3\otimes x_1 \\ \varepsilon x_1\otimes x_2 & \varepsilon x_2\otimes x_2 & q_{12}\, x_3\otimes x_2 \\ q_{21}\, x_1\otimes x_3 & q_{21}(x_2+x_1)\otimes x_3 & q_{22}\, x_3\otimes x_3 \end{pmatrix}$$

§8.14. The block has $\varepsilon=-1$

Here $\mathfrak{B}(V_1)\simeq \Lambda(V_1)$ is an exterior algebra

(8.14) $g_2 z_1 = q_{21}q_{22}(z_1+w_1)$, $\partial_3(z_1) = (1-\tilde{q}_{12})x_2 - \tilde{q}_{12}x_1$

(8.15) $\delta(z_1) = g_1 g_2\otimes z_1 + ((1-\tilde{q}_{12})x_2 - \tilde{q}_{12}x_1)\, g_2\otimes x_3$

Case 1: $\tilde{q}_{12}=1$

Case 2: $\tilde{q}_{12}\neq 1$ $\quad(\tilde{q}_{12}=q_{12}q_{21})$

The braided V.S. V^{diag} has Dynkin diagram

$\overset{-1}{\circ}\xrightarrow{\tilde{q}_{12}}\overset{q_{22}}{\circ}\xrightarrow{\tilde{q}_{12}}\overset{-1}{\circ}$ $\quad V^{diag} := \operatorname{gr} V$ $\quad x_1 x_3,\ J_i \cdot x_i\ f^1$

By Hypothesis 1.7

and after inspection of the Tables 2 in [H2]

⇒ there are 4 possible diagrams, listed next:

① $q_{22}=-1=\tilde{q}_{12}$ Tab 2 ⑰ lem 8.8 GK = 2

② $q_{22}=q$, $\tilde{q}_{12}=q^{-1}$, $q^2\neq 1$, Tab 2 ⑧, lem 8.10 GK = ∞

③ $q_{22}=-1$, $\tilde{q}_{12}=w\in G_3'$, Tab 2 ⑱, lem 8.9 ∞

④ $q_{22}=-w$, $\tilde{q}_{12}=w\in G_3'$, Tab 2 ⑮, lem 8.11 ∞

For $q\in k^\times$, let $\mathfrak{C}_*(q)=V$ be the braided V.S.

as in (8.3), under the assumptions that

$\varepsilon=-1$, $q_{22}=-1$, $q_{12}=q$, $q_{21}=-q^{-1}$

Prop 8.8. The algebra $\mathfrak{B}(\mathfrak{C}_*(q))$ is presented by generators x_1, x_2, x_3 and relations (8.17)

$(x_1^2=0,\ x_2^2=0,\ x_1x_2=-x_2x_1)$

(8.25) $x_3^2=0$, $x_{31}^2=0$

(8.26) $x_2[x_{23},x_{13}]_c - q^2[x_{23},x_{13}]_c\, x_2 = q\, x_{13}x_{213}$,

(8.27) $x_{213}^2=0$

$(x_{ij}=(\mathrm{ad}_c x_i)\, x_j$, $x_{i_1 i_2\cdots i_m}=(\mathrm{ad}_c x_{i_1})\, x_{i_2\cdots i_m})$

Moreover $\mathfrak{B}(\mathfrak{C}_*(q))$ has a PBW-basis

$B=\{x_2^{m_1}x_{23}^{m_2}x_{213}^{m_3}[x_{23},x_{13}]_c^{m_4}x_1^{m_5}x_{13}^{m_6}x_3^{m_7}$

$m_1, m_3, m_5, m_6, m_7\in\{0,1\},\ m_2, m_4\in\mathbb{N}_0\}$

∴ $\mathrm{GK}\dim(\mathfrak{B}(\mathfrak{C}_*(q)))=2$

Pf: ① ∵ $\mathfrak{B}(V_1)\simeq\Lambda(V_1)$, ∴ $x_1^2=0$, $x_2^2=0$, $x_1x_2=-x_2x_1$

∵ x_1, x_3 generates a Nichols algebra of Cartan type A_2 at -1

∴ $x_3^2=0$, $x_{31}^2=0$ (8.25)

② $[x_{23},x_{13}]_c = x_{23}x_{13}+x_{13}x_{23}$ $\quad\mathrm{ad}_c$ — primitive

then (8.28) $x_2x_{23} = -q\, x_{23}x_2$,

$x_2x_{213} = q\, x_{213}x_2$,

(8.29) $x_{23}x_1 = -q^{-1}x_1x_{23} - q^{-1}x_{213}$

$x_{23}x_3 = ((\mathrm{ad}_c x_2)x_3)x_3 = (x_2x_3 - q\, x_3x_2)x_3$

$= x_2x_3^2 - q\, x_3x_2x_3 = -q\, x_3x_2x_3$

$-q\, x_3x_{23} = -q\, x_3\,\mathrm{ad}_c x_2(x_3) = -q\, x_3(x_2x_3 - q\, x_3x_2)$

$= -q\, x_3x_2x_3$

$x_{213} = -x_1x_{23} - q\, x_{23}x_1$

① right $= -x_1(x_2x_3 - q\, x_3x_2) - q(x_2x_3 - q\, x_3x_2)x_1$

$= -x_1x_2x_3 + q\, x_1x_3x_2 - q\, x_2x_3x_1 + q^2 x_3x_2x_1$

② left $= x_{213} = \mathrm{ad}_c x_2(\mathrm{ad}_c x_1(x_3))$

$= \mathrm{ad}_c x_2(x_1x_3 - q\, x_3x_1)$

$= x_2x_1x_3 + x_1x_2x_3 - q\, x_2x_3x_1 + q^2x_3x_2x_1$

$\mathrm{ad}_c x_2(x_1x_3) = x_{2(-1)}\cdot x_1x_3$

$\mathrm{ad}_c x_2(x_1x_3) = x_2x_1x_3 - \mu(c(x_2\otimes x_1x_3))$

$= x_2x_1x_3 - \mu(x_{2(-1)}\cdot x_1x_3\otimes x_{2(0)})$, $\quad x_{2(-1)} = g_1$, $x_{2(0)} = x_2$

$g_1\cdot(x_1x_3)$

$=(g_1\cdot x_1)(g_1\cdot x_3)$ $\quad = x_2x_1x_3 + q\, x_1x_3x_2$

$= -q_{12}x_1x_3$

(8.30) ~ (8.32)

③ Now we prove (8.26) holds in $\mathfrak{B}(V)$

$\partial_3(x_2[x_{23},x_{13}]_c - q^2[x_{23},x_{13}]_c\, x_2)$

$= 2q^{-1}x_2x_1x_{13} + 2x_1x_{13}(x_2+x_1)$

$= -2q^{-1}x_1(x_{213} - q\, x_{13}x_2) + 2x_1x_{13}x_2$

$= -2q^{-1}x_1x_{213} + 4x_1x_{13}x_2$

$\shortparallel$

$\partial_3(x_{13}x_{213})$

④ for (8.27) check $\partial_i(x_{213}^2)=0$, for $i=1,2,3$

∵ ∂_1, ∂_2 annihilate x_{213}

∴ it remains the case $i=3$

$\partial_3(x_{213}^2) = 4x_2x_1(g_3\cdot x_{213}) + 4x_{213}x_2x_1$

$= 4(q^2x_2x_1x_{213} + x_{213}x_2x_1) = 0$

⑤ Hence the quotient $\tilde{\mathfrak{B}}$ of $T(\mathfrak{C}_*(q))$ by (8.17), (8.25), (8.26) and (8.27) projects on to $\mathfrak{B}(\mathfrak{C}_*(q))$

∵ (8.28) ~ (8.33) holds in $\tilde{\mathfrak{B}}$,

∴ $x_{213}x_{23} = q(x_{23}+x_{13})x_{213}$

$x_{213}[x_{23},x_{13}]_c = q^2[x_{23},x_{13}]_c\, x_{213}$

$[x_{23},x_{13}]_c\, x_{23} = (x_{23}+x_{13})[x_{23},x_{13}]_c$

Thus the subspace I spaned by B is a right ideal of $\tilde{\mathfrak{B}}$

⑥ $\tilde{\mathfrak{B}}\simeq\mathfrak{B}(\mathfrak{C}_*(q))$

B is linealy independent in $\mathfrak{B}(\mathfrak{C}_*(q))$

$((8.4)\ V_1 = \langle x_1, x_3\rangle\subset V_2 = V)$

We claim that the classes of $[x_{23},x_{13}]_c$ and x_{23}^2 in $\mathfrak{B}^{diag}$ are non-zero primitive elements of $\mathfrak{B}^{diag}$

$\delta([x_{23},x_{13}]_c) = [x_{23},x_{13}]_c\otimes 1 - 2x_{13}x_1\otimes x_3 - x_{13}\otimes x_{13} - 2x_1\otimes x_3x_{13} + 1\otimes[x_{23},x_{13}]_c$

$\delta(x_{23}^2) = x_{23}^2\otimes 1 - x_{13}\otimes x_{23} + (2q^{-1}x_2x_{13} - q\, x_1x_{13})\otimes x_3 - 2x_1\otimes x_{23}x_3 + 1\otimes x_{23}^2$

$\mathfrak{B}(V)$

∴ to show $[x_{23},x_{13}]_c\in\mathfrak{B}_5^4 - \mathfrak{B}_4^4$ $\quad$ P21

$x_{23}^2\in\mathfrak{B}_6^4 - \mathfrak{B}_5^4$

P21 / Rmk 3.6 (a) $\operatorname{gr}\mathfrak{B} = \bigoplus_{i\in\mathbb{N}_0}\operatorname{gr}^i\mathfrak{B}$

where $\operatorname{gr}^i\mathfrak{B} = \mathfrak{B}_i/\mathfrak{B}_{i-1}$ is a graded Hopf algebra in ${}^H_H\mathcal{YD}$

(c) $\mathfrak{B} = \bigoplus_{n\in\mathbb{N}_0}\mathfrak{B}^n$, $\mathfrak{B}_i = \bigoplus_{n\in\mathbb{N}_0}\mathfrak{B}_i^n$

x_1, x_3 一次，不提和 $x_{23}x_3$

(7.1) Suppose $[x_{23},x_{13}]_c\in\mathfrak{B}_4^4$

∵ $\mathfrak{B}(V)$ is spanned by B

there exists $a\in k$, s.t. $[x_{23},x_{13}]_c = a\, x_1x_{13}x_3$,

$0 = -q^2a\, x_{13}x_3$, $\quad 2q^{-1}x_1x_{13} = a\, x_1x_{13}$

∴ contradiction

∴ $[x_{23},x_{13}]_c\in\mathfrak{B}_5^4 - \mathfrak{B}_4^4$

(7.2) $x_{23}^2\in\mathfrak{B}_5^4$

∴ $\exists a_i\in k$, s.t. $x_{23}^2 = a_1[x_{23},x_{13}]_c + a_2x_{23}x_{13} + \cdots a_8x_2x_{13}x_3$

∴ $0 = \partial_2(x_{23}^2) = q\, a_7x_1x_{13} - q^2a_8x_{13}x_3$

∴ $a_7 = a_8 = 0$

∂_1: $0 = q\, a_3x_{23}x_3 + a_5x_{213} - q^2a_6x_{13}x_3$

∴ $a_3 = a_5 = a_6 = 0$

∂_3: $2q^{-1}x_2x_{13} - q\, x_1x_{13} = 2q^{-1}a_1x_1x_{13} + \cdots a_4x_{213}$

∴ $2a_2=0$, $2q^{-1}a_2 = 2q^{-1}$

∴ contradiction

∴ $x_{23}^2\in\mathfrak{B}_6^4 - \mathfrak{B}_5^4$

⑧ ∵ V^{diag} is of Type A_3 at $q=-1$ $\quad$ A_3 正根

∴ $\mathfrak{B}^{diag}$ has a PBW basis

whose set of generators contains x_2, x_{23}, x_{213}, x_1, x_{13}, x_3 $\quad$ $\alpha_1+\alpha_3$, α_2+

$[x_{23},x_{13}]_c$

x_{23} and $[x_{23},x_{13}]$ have infinitely height in $\mathfrak{B}^{diag}$

$\cancel{x_{23}x_{23}} = g_1g_2$ 双特征

GK-dim

Lem 8.9 $\quad K = \mathfrak{B}(V)^{\mathrm{co}\,\mathfrak{B}(W)}$

$K^1 = \mathrm{ad}_c$, $\quad W = \langle x_3, z_3\rangle$

$W \simeq V(w, 2)$

Lem 8.10, $\quad u = [x_{23}, x_3]_c$ $\quad u\in\mathfrak{B}_4^3 - ?$

∴ $\mathrm{GKdim}\,\mathfrak{B}(q)\geq$ — $\tilde{B}_0$ — $\tilde{B}_1$ → $\mathrm{GKdim}\,\mathfrak{B}(W)$